MULTIPLE INTEGRALS - TASKS (I PART)

Double integrals-solving

Example 1.

Integrate $\iint\limits_D \frac{x^2}{1+y^2}dxdy$ if $0 \le x \le 1$ and $0 \le y \le 1$

Solution :

Here we are immediately given the limits of integrals, so we do not have to draw a picture and determine them. It is always a question whether it is easier to work first "by x" and then "by y" or vice versa … Consider a little, and then start working …

$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int_0^1 dy \int_0^1 \frac{x^2}{1+y^2}dx$ Some professors prefer to write as follows:

$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2}dx \right) dy$ You of course work as required by your professor ….

So, first solve the integral in brackets. He is "by x" and ***y*** are treated here as a constant!

$$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2}dx \right) dy = \int_0^1 \frac{1}{1+y^2} \left(\int_0^1 x^2 dx \right) dy =$$

We'll solve it "on the side" ….

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

Now this solution is reintroduced into the double integral:

$$\iint\limits_D \frac{x^2}{1+y^2}dxdy = \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2}dx \right) dy = \int_0^1 \frac{1}{1+y^2} \left(\int_0^1 x^2 dx \right) dy = \int_0^1 \frac{1}{1+y^2}dy =$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1+y^2}dy = \frac{1}{3} arctgy\Big|_0^1 = \frac{1}{3}\left(arctg1 - arctg0\right) = \frac{1}{3}(\frac{\pi}{4} - 0) = \boxed{\frac{\pi}{12}}$$

And this is solution.

Example 2.

Integrate $\iint_D x\sin(x+y)dxdy$ if $0 \le x \le \pi$ and $0 \le y \le \frac{\pi}{2}$

Solution:

$$\iint_D x\sin(x+y)dxdy = \int_0^{\pi} xdx \int_0^{\frac{\pi}{2}} \sin(x+y)dy = \int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}} \sin(x+y)dy\right)dx$$

The integral in brackets we will solve "on the side" :

$$\int_0^{\frac{\pi}{2}} \sin(x+y)dy = -\cos(x+y)\Bigg|_0^{\frac{\pi}{2}} = -[\cos(x+\frac{\pi}{2}) - \cos(x+0)] = -[\cos(x+\frac{\pi}{2}) - \cos x]$$

From trigonometry we know that $\cos(x+\frac{\pi}{2}) = -\sin x$, so:

$$\int_0^{\frac{\pi}{2}} \sin(x+y)dy = -\cos(x+y)\Bigg|_0^{\frac{\pi}{2}} = -[\cos(x+\frac{\pi}{2}) - \cos(x+0)] = -[\cos(x+\frac{\pi}{2}) - \cos x] = -[-\sin x - \cos x]$$

$$= \sin x + \cos x$$

Let us return to the double integral:

$$\iint_D x\sin(x+y)dxdy = \int_0^{\pi} xdx \int_0^{\frac{\pi}{2}} \sin(x+y)dy = \int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}} \sin(x+y)dy\right)dx = \int_0^{\pi} x(\sin x + \cos x)dx$$

Here we have a partial integration for both integrals :

$$\int_0^{\pi} x\sin xdx = \begin{vmatrix} x = u & \sin xdx = dv \\ dx = du & -\cos x = v \end{vmatrix} = -x\cos x - \int(-\cos x)dx =$$

$$= \left(-x\cos x + \sin x\right)\Big|_0^{\pi} = \left(-\pi\cos\pi + \sin\pi\right) - \left(-0\cos 0 + \sin 0\right) = -\pi(-1) = \pi$$

$$\int_0^{\pi} x\cos x dx = \begin{vmatrix} x = u & \cos x dx = dv \\ dx = du & \sin x = v \end{vmatrix} = x\sin x - \int \sin dx =$$

$$= (x\sin x + \cos x)\Big|_0^{\pi} = (\pi\sin\pi + \cos\pi) - (0\sin 0 + \cos 0) = -1 - 1 = -2$$

The solution will be:

$$\iint_D x\sin(x+y)dxdy = \int_0^{\pi} xdx \int_0^{\frac{\pi}{2}} \sin(x+y)dy = \int_0^{\pi} x\left(\int_0^{\frac{\pi}{2}} \sin(x+y)dy\right)dx = \int_0^{\pi} x(\sin x + \cos x)dx = \boxed{\pi - 2}$$

As you view this double integral, we first solve “by y” and then “by x”.

Would it be easier to have gone the other way around?

Let's see:

II method

$$\iint_D x\sin(x+y)dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^{\pi} x\sin(x+y)dx\right)dy$$

We will solve the integral in brackets to the side, as vague, and we will add the border:

$$\int_0^{\pi} x\sin(x+y)dx = ?$$

$$\int x\sin(x+y)dx = \begin{vmatrix} x = u & \sin(x+y)dx = dv \\ dx = du & -\cos(x+y) = v \end{vmatrix} = -x\cos(x+y) - \int[-\cos(x+y)]dx =$$

$$= -x\cos(x+y) + \sin(x+y)$$

$$\int_0^{\pi} x\sin(x+y)dx = -x\cos(x+y) + \sin(x+y)\Big|_0^{\pi} = [-\pi\cos(\pi+y) + \sin(\pi+y)] - [-0\cos(0+y) + \sin(0+y)] =$$

$$= -\pi\cos(\pi+y) + \sin(\pi+y) - \sin y$$

$$\iint_D x\sin(x+y)dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^{\pi} x\sin(x+y)dx\right)dy = \int_0^{\frac{\pi}{2}} \left(-\pi\cos(\pi+y) + \sin(\pi+y) - \sin y\right)dy =$$

$$= \left(-\pi\sin(\pi+y) - \cos(\pi+y) + \cos y\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \left(-\pi\sin(\pi+\frac{\pi}{2}) - \cos(\pi+\frac{\pi}{2}) + \cos\frac{\pi}{2}\right) - \left(-\pi\sin(\pi+0) - \cos(\pi+0) + \cos 0\right) =$$

$$= \pi - 0 + 0 - (0 + 1 + 1) = \boxed{\pi - 2}$$

Maybe a little faster ... It is essential that the solution is correct!

Example 3.

Integrate $\iint\limits_D xy^2dxdy$ if the area of integration is limited with parabola $y^2 = 2x$ and line $x = \frac{1}{2}$.

Solution:

First, we will determine the sections and draw a picture:

The intersection is determined by solving the equation system:

$y^2 = 2x$

$x = \frac{1}{2}$

$y^2 = 2\cdot\frac{1}{2} \rightarrow y^2 = 1 \rightarrow y = \pm 1 \rightarrow M(\frac{1}{2},-1) \wedge N(\frac{1}{2},1)$

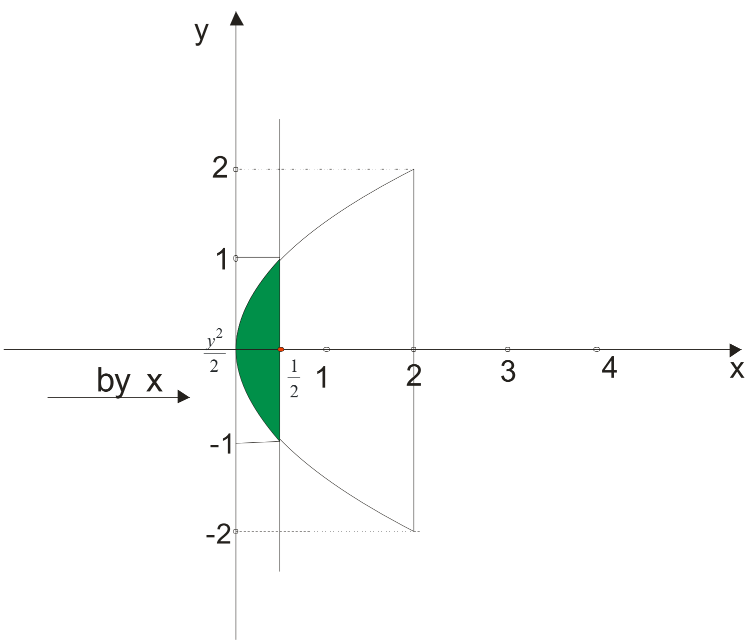


Image helps us to determine area of integration $D:\begin{cases} -1 \le y \le 1 \\ \frac{y^2}{2} \le x \le \frac{1}{2} \end{cases}$ (See previous file)

Now we have:

$$\iint\limits_D xy^2dxdy = \int_{-1}^{1} dy \int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2dx = \int_{-1}^{1}\left(\int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2dx \right)dy$$ First solve the integral in brackets….

$$\int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2dx = y^2 \cdot \int_{\frac{y^2}{2}}^{\frac{1}{2}} xdx = y^2 \cdot [\frac{x^2}{2}]\Bigg|_{\frac{y^2}{2}}^{\frac{1}{2}} = y^2 \cdot [\frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{y^2}{2}\right)^2}{2}] = \frac{y^2}{2}[\frac{1}{4} - \frac{y^4}{4}] = \frac{1}{8}[y^2 - y^6]$$

We return to the double integral:

$$\iint_D xy^2dxdy = \int_{-1}^{1} dy \int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2dx = \int_{-1}^{1}\left(\frac{1}{8}[y^2 - y^6]\right)dy =$$

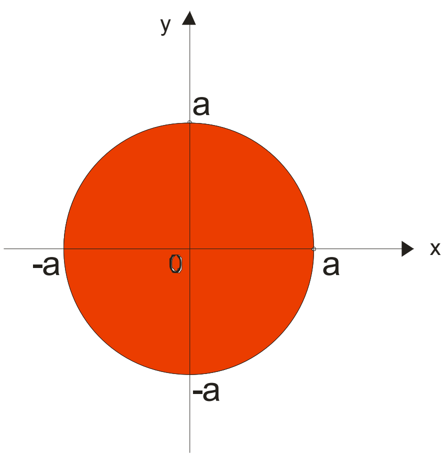
$$= \frac{1}{8}[\frac{y^3}{3} - \frac{y^7}{7}]\Bigg|_{-1}^{1} = \frac{1}{8}\{[\frac{1^3}{3} - \frac{1^7}{7}] - [\frac{(-1)^3}{3} - \frac{(-1)^7}{7}]\} = \frac{1}{8} \cdot \frac{8}{21} = \boxed{\frac{1}{21}}$$

Example 4.

Find $\iint_D \sqrt{x^2 + y^2}\,dxdy$ if area D is given with $x^2 + y^2 \le a^2$

Solution:

Let's look at the picture:



In such cases, when circle is given, it is convenient to switch to polar coordinates:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$|J| = r$$

And then solve the integral:

$$\iint_D z(x,y)dxdy = \iint_{D`} z(r\cos\varphi, r\sin\varphi)|J|drd\varphi = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(r\cos\varphi, r\sin\varphi)rdr$$

$x^2 + y^2 = a^2$

$(r\cos\varphi)^2 + (r\sin\varphi)^2 = a^2$

$r^2(\cos^2\varphi + \sin^2\varphi) = a^2$ We know that $\cos^2\varphi + \sin^2\varphi = 1$, so:

$r^2 = a^2 \to r = a$

So: r goes from 0 to a.

It is clear that $0 \le \varphi \le 2\pi$

We have D`$= \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$ and then :

$$\iint_D \sqrt{x^2 + y^2}\,dxdy = \int_0^{2\pi} d\varphi \int_0^a \sqrt{r^2}\,rdr = \int_0^{2\pi}\left(\int_0^a \sqrt{r^2}\,rdr\right)d\varphi = \int_0^{2\pi}\left(\int_0^a r^2dr\right)d\varphi =$$

As always, we solve the integral in brackets:

$$\int_0^a r^2dr = \frac{r^3}{3}\Big|_0^a = \frac{a^3}{3}$$

Now, we have :

$$\iint_D \sqrt{x^2 + y^2}\,dxdy = \int_0^{2\pi} d\varphi \int_0^a \sqrt{r^2}\,rdr = \int_0^{2\pi}\left(\int_0^a \sqrt{r^2}\,rdr\right)d\varphi = \int_0^{2\pi}\left(\int_0^a r^2dr\right)d\varphi =$$

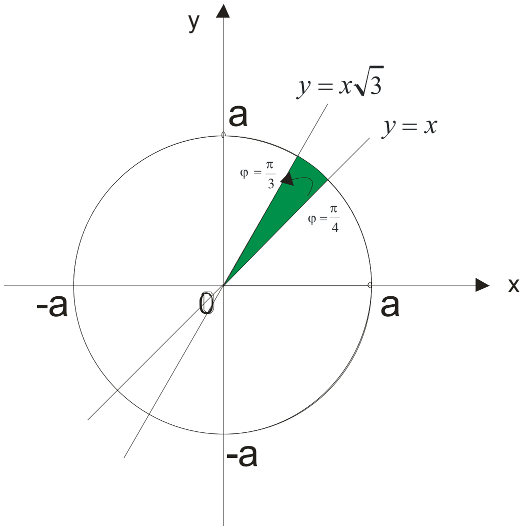
$$= \int_0^{2\pi} \frac{a^3}{3}d\varphi = \frac{a^3}{3}\int_0^{2\pi} d\varphi = \frac{a^3}{3}\cdot\varphi\Big|_0^{2\pi} = \frac{a^3}{3}(2\pi - 0) = \boxed{\frac{2\pi a^3}{3}}$$

Example 5.

Calculate $\iint_D \sqrt{a^2 - x^2 - y^2}\,dxdy$ if area D is limited with $x^2 + y^2 = a^2$, $y = x, y = x\sqrt{3}$ in first quadrant.

Solution:

Picture:



We will use polar coordinates:

$x = r\cos\varphi$

$y = r\sin\varphi$

$|J| = r$

$x^2 + y^2 = a^2$

$(r\cos\varphi)^2 + (r\sin\varphi)^2 = a^2$

$r^2(\cos^2\varphi + \sin^2\varphi) = a^2$

$r^2 = a^2 \rightarrow r = a$

So : r goes from 0 to a.

From lines $y = x, y = x\sqrt{3}$ we will determine borders for φ

To recall:

Line $y = kx + n$ has direction coefficient $k = tg\varphi$.

From $y = x$ is $k=1$, so $tg\varphi = 1 \rightarrow \varphi = \frac{\pi}{4}$

From $y = x\sqrt{3}$ is $k = \sqrt{3}$, so $tg\varphi = \sqrt{3} \rightarrow \varphi = \frac{\pi}{3}$

So we get that:

$$D` = \begin{cases} 0 \le r \le a \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \end{cases}$$

Now we solve the integral:

$$\iint_D \sqrt{a^2 - x^2 - y^2}\,dxdy = \iint_D \sqrt{a^2 - (x^2 + y^2)}\,dxdy = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^a \sqrt{a^2 - r^2} \cdot rdr$$

$$\int_0^a \sqrt{a^2 - r^2} \cdot rdr \quad = ?$$

$$\int \sqrt{a^2 - r^2} \cdot rdr = \left| \begin{matrix} a^2 - r^2 = t^2 \\ -2rdr = 2tdt \\ rdr = -tdt \end{matrix} \right| = \int \sqrt{t^2}(-t)dt = -\int t^2 dt = -\frac{t^3}{3} = -\frac{\left(\sqrt{a^2 - r^2}\right)^3}{3}$$

$$\int_0^a \sqrt{a^2 - r^2} \cdot rdr = -\frac{\left(\sqrt{a^2 - r^2}\right)^3}{3} \Bigg|_0^a = -[\frac{\left(\sqrt{a^2 - a^2}\right)^3}{3} - \frac{\left(\sqrt{a^2 - 0^2}\right)^3}{3}] = -[-\frac{a^3}{3}] = \frac{a^3}{3}$$

$$\iint_D \sqrt{a^2 - x^2 - y^2}\,dxdy = \iint_D \sqrt{a^2 - (x^2 + y^2)}\,dxdy = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^a \sqrt{a^2 - r^2} \cdot rdr =$$

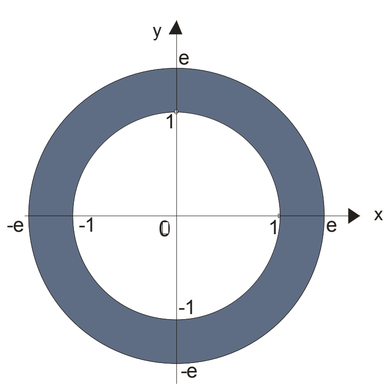
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a^3}{3} d\varphi = \frac{a^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi = \frac{a^3}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{a^3}{3}\frac{\pi}{12} = \boxed{\frac{a^3\pi}{36}}$$

Example 6.

Calculate $\iint_D \frac{\ln(x^2 + y^2)}{x^2 + y^2} dxdy$ if area D is between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = e^2$

<u>*Solution:*</u>

Picture:



We take polar coordinates:

$$x = r\cos\varphi \qquad x^2+y^2=1 \qquad x^2+y^2=e^2$$
$$y = r\sin\varphi \qquad (r\cos\varphi)^2+(r\sin\varphi)^2=1 \qquad (r\cos\varphi)^2+(r\sin\varphi)^2=e^2$$
$$|J| = r \qquad r^2(\cos^2\varphi+\sin^2\varphi)=1 \qquad r^2(\cos^2\varphi+\sin^2\varphi)=e^2$$
$$r^2=1 \to r=1 \qquad r^2=e^2 \to r=e$$

So, we have $1 \le r \le e$

And $0 \le \varphi \le 2\pi$.

$$\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^{\not 2}} \cdot \not r\, dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2\ln r}{r} dr$$

$$\int_1^e \frac{2\ln r}{r}dr = 2\int_1^e \frac{\ln r}{r}dr$$

$$\int \frac{\ln r}{r}dr = \begin{vmatrix} \ln r = t \\ \frac{1}{r}dr = dt \end{vmatrix} = \int t\,dt = \frac{t^2}{2} = \frac{\ln^2 r}{2}$$

$$\int_1^e \frac{2\ln r}{r}dr = 2\int_1^e \frac{\ln r}{r}dr = \not 2 \frac{\ln^2 r}{\not 2}\Big|_1^e = \ln^2 e - \ln^2 1 = 1-0 = 1$$

$$\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^{\not 2}} \cdot \not r\, dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2\ln r}{r}dr = \int_0^{2\pi} d\varphi = 2\pi - 0 = \boxed{2\pi}$$

Example 7.

Find $\iint_D xydxdy$,where area D is limited with Ox axis and circles arcs:

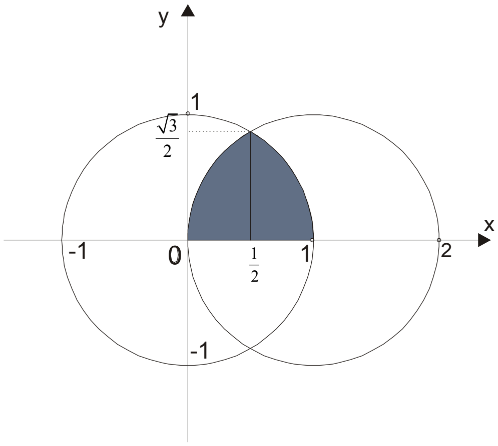
$x^2+y^2=1$ and $x^2+y^2-2x=0$ in the first quadrant.

Solution:

First to "pack" a circle, then find sections and draw a picture:

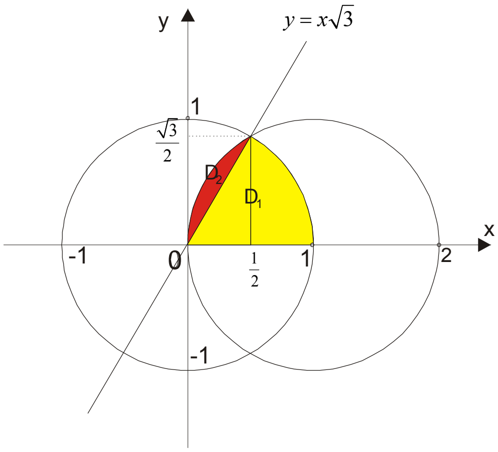
$$x^2+y^2-2x=0$$
$$x^2-2x+1-1+y^2=0$$
$$(x-1)^2+y^2=1$$

The intersection is : $x^2+y^2-2x=0 \wedge x^2+y^2=1 \to 1-2x=0 \to x=\frac{1}{2} \to y=\frac{\sqrt{3}}{2}$



This area must be divided into two parts:

Line that passes through the point of intersection circles and point O(0,0) is $y=\sqrt{3}x$



Again we pass to polar coordinates: $\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ |J| &= r \end{aligned}$

For area D_1 is:

$$x^2+y^2=1$$
$$(r\cos\varphi)^2+(r\sin\varphi)^2=1$$
$$r^2(\cos^2\varphi+\sin^2\varphi)=1$$
$$r^2=1\to r=1$$

So: $0\le r\le 1$

The angle goes from x- axis to line $y=\sqrt{3}x$, so: $0\le\varphi\le\frac{\pi}{3}$

For area D_2 we have :

$$x^2 + y^2 - 2x = 0$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 - 2r\cos\varphi = 0$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = 2r\cos\varphi$$
$$r^2 = 2r\cos\varphi \to r = 2\cos\varphi$$

So: $0 \le r \le 2\cos\varphi$

The angle goes from line $y = \sqrt{3}x$ to y – axes , so : $\frac{\pi}{2} \le \varphi \le \frac{\pi}{3}$

We have:

$$D_1 : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{3} \end{cases} \qquad D_2 : \begin{cases} 0 \le r \le 2\cos\varphi \\ \frac{\pi}{2} \le \varphi \le \frac{\pi}{3} \end{cases}$$

As is:

$$xy = r\cos\varphi \cdot r\sin\varphi = r^2 \sin\varphi\cos\varphi$$

integral will be:

$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin\varphi\cos\varphi \cdot rdr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \sin\varphi\cos\varphi \cdot rdr =$$
$$= \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3 \sin\varphi\cos\varphi dr$$

Each will be solved and then we gather solution:

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr = \int_0^{\frac{\pi}{3}} \sin\varphi\cos\varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} s\text{in}\,\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Bigg|_0^1 = \frac{1}{4}\int_0^{\frac{\pi}{3}} s\text{in}\,\varphi\cos\varphi d\varphi$$

$$\int s\text{in}\,\varphi\cos\varphi d\varphi = \begin{vmatrix} \sin\varphi = t \\ \cos\varphi d\varphi = dt \end{vmatrix} = \int tdt = \frac{t^2}{2} = \frac{\sin^2\varphi}{2}$$

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin\varphi\cos\varphi dr = \int_0^{\frac{\pi}{3}} \sin\varphi\cos\varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} s\text{in}\,\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Bigg|_0^1 = \frac{1}{4}\frac{\sin^2\varphi}{2}\Bigg|_0^{\frac{\pi}{3}} =$$

$$= \frac{1}{8}\left(\sin^2\frac{\pi}{3} - \sin^2 0\right) = \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{8}\left(\frac{3}{4}\right) = \boxed{\frac{3}{32}}$$

Now the second integral:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3 \sin\varphi\cos\varphi dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \int_0^{2\cos\varphi} r^3 dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \cdot \frac{r^4}{4}\Bigg|_0^{2\cos\varphi} =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos\varphi \cdot \frac{16\cos^4\varphi}{4} d\varphi = 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos^5\varphi d\varphi =$$

First :

$$\int \sin\varphi\cos^5\varphi d\varphi = \begin{vmatrix} \cos\varphi = t \\ -\sin\varphi d\varphi = dt \end{vmatrix} = -\int t^5 dt = -\frac{t^6}{6} = -\frac{\cos^6\varphi}{6} \text{ and now:}$$

$$4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\varphi\cos^5\varphi d\varphi = 4\left(-\frac{\cos^6\varphi}{6}\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -4\left[\frac{\cos^6\frac{\pi}{2}}{6} - \frac{\cos^6\frac{\pi}{3}}{6}\right] = -4\left[0 - \frac{\frac{1}{64}}{6}\right] = \boxed{\frac{1}{96}}$$

Finally:

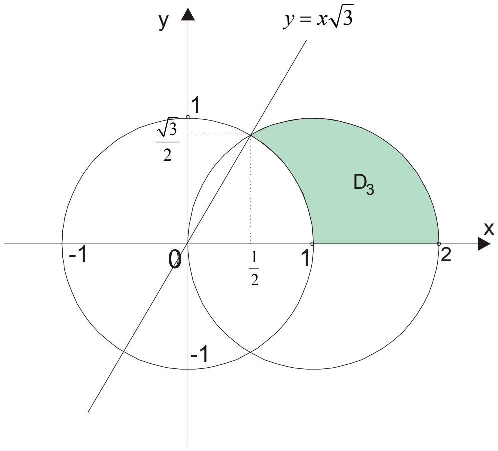
$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin\varphi\cos\varphi \cdot rdr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \sin\varphi\cos\varphi \cdot rdr = \frac{3}{32} + \frac{1}{96} = \boxed{\frac{5}{48}}$$

And this would be the solution of our task. **BUT!**

WE have overlooked one thing!

Here there is another possible area!!

Let's look at the picture again:



And this area is limited given circles and x axis in the first quadrant!

Watch out for this, the task can be in two parts and that you are not tips by Professor

There we have:

$$D_3 : \begin{cases} 1 \le r \le 2\cos\varphi \\ 0 \le \varphi \le \dfrac{\pi}{3} \end{cases}$$

$$\iint_D xydxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2\cos\varphi} r^2 \sin\varphi \cos\varphi \cdot rdr = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2\cos\varphi} r^3 \sin\varphi \cos\varphi dr = \frac{9}{16}$$

Example 8.

Calculate the value of the integral $\iint_D (y-x)dxdy$ if area is given with lines:

$y = x + 1$

$y = x - 3$

$y = -\frac{1}{3}x + \frac{7}{3}$

$y = -\frac{1}{3}x + 5$

putting that is:
$$u = y - x$$
$$v = y + \frac{1}{3}x$$

Solution:

To remind:

If x=x(u,v) and y=y(u,v)

$$J=\frac{D(x,y)}{D(u,v)} \neq 0$$

Formula is:

$$\iint_D z(x,y)dxdy = \iint_{D`} z[x(u,v),y(u,v)]|J|dudv$$

Here we first referred to express the x and y:

$$u = y - x$$
$$v = y + \frac{1}{3}x ..../*3$$

$$u = y - x$$
$$3v = 3y + x$$

$$u + 3v = 4y \rightarrow \boxed{y = \frac{1}{4}u + \frac{3}{4}v}$$

$$u = y - x........................*(-3)$$
$$3v = 3y + x$$

$$-3u = -3y + 3x$$
$$3v = 3y + x$$

$$-3u + 3v = 4x \rightarrow \boxed{x = -\frac{3}{4}u + \frac{3}{4}v}$$

Now we ask for the Jacobian:

$$\left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = \left|-\frac{9}{16} - \frac{3}{16}\right| = \left|-\frac{12}{16}\right| = \left|-\frac{3}{4}\right| = \frac{3}{4}$$

So: $|J| = \frac{3}{4}$

How to set boundaries for u and v?

$$\left.\begin{array}{l} y = x+1 \\ y = x-3 \end{array}\right\} \to \left.\begin{array}{l} y-x=1 \\ y-x=-3 \end{array}\right\} \to \left.\begin{array}{l} u=1 \\ u=-3 \end{array}\right\} \to \boxed{-3 \le u \le 1}$$

$$\left.\begin{array}{l} y = -\frac{1}{3}x+\frac{7}{3} \\ y = -\frac{1}{3}x+5 \end{array}\right\} \to \left.\begin{array}{l} y+\frac{1}{3}x=\frac{7}{3} \\ y+\frac{1}{3}x=5 \end{array}\right\} \to \left.\begin{array}{l} v=\frac{7}{3} \\ v=5 \end{array}\right\} \to \boxed{\frac{7}{3} \le v \le 5}$$

Now, we have:

$$\iint_D (y-x)dxdy = \iint_{D`}\left( (\frac{1}{4}u+\frac{3}{4}v)-(-\frac{3}{4}u+\frac{3}{4}v) \right)\cdot\frac{3}{4}dudv = \iint_{D`}\left(\frac{1}{4}u+\cancel{\frac{3}{4}v}+\frac{3}{4}u-\cancel{\frac{3}{4}v} \right)\cdot\frac{3}{4}dudv =$$

$$= \iint_{D`} u\cdot\frac{3}{4}dudv = \int_{\frac{7}{3}}^{5}\left(\frac{3}{4}\int_{-3}^{1} udu \right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}\left(\frac{u^2}{2}\bigg|_{-3}^{1} \right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}\left(\frac{1^2}{2}-\frac{(-3)^2}{2} \right)dv = \frac{3}{4}\int_{\frac{7}{3}}^{5}(-4)dv =$$

$$= -3\int_{\frac{7}{3}}^{5} dv = -3v\bigg|_{\frac{7}{3}}^{5} = -3(5-\frac{7}{3}) = -3\cdot\frac{8}{3} = \boxed{-8}$$